ON FACTORIZATION OF MEROMORPHIC FUNCTIONS(1)

BY
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1. **Introduction.** A meromorphic function $h(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is nonlinear and meromorphic and $g(z)$ is nonlinear and entire (g may be meromorphic when $f(z)$ is rational). $h(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear (a polynomial or $f(z)$ is rational).

There are numerous questions that one can ask about factorization of meromorphic functions. We shall primarily be concerned with two of them.

1. How many factors does a given meromorphic function have?

2. Given certain properties of a meromorphic function, what are some related properties of its factors?

2. **Generalizations and extensions of previous results.** We begin with the simple

THEOREM 1. *Any transcendental meromorphic function of finite order which has at most a finite number of poles and zeros is pseudo-prime.*

We shall need the following lemma.

LEMMA 1 (EDREI AND FUCHS [2]). *If f is any meromorphic function and g is entire, then $f(g)$ is of finite order implies that either f is of finite order and g is a polynomial or that f is of zero order.*

Proof of theorem. Let $h=f(g)$ satisfy the hypotheses of the theorem and suppose that it is not pseudo-prime. Clearly $f(z)$ has at most one pole, say b, and at most one zero, say a. Thus it can be expressed as

$$f(z)=[(z-a)^n/(z-b)^m]e^{\alpha(z)},$$

where n and m are nonnegative integers and $\alpha(z)$ is entire. By Lemma 1, $\alpha(z)$ must be a constant and the proof is complete.

We note that $f(g)$ must have either infinitely many poles or zeros unless one of n, m is zero, so that $f(z)$ is of the form $c(z-a)^n$, where n is an integer and c is a constant.

This generalizes a result of Thron [11].

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Along these same lines we have

THEOREM 2. *Let h be an entire (meromorphic) function with at most a finite number of simple zeros. Either h has only polynomial right factors or every left factor has at most 2 (4) simple zeros.*

Proof. We use the standard notations of the Nevanlinna Theory. In particular $N(r, a; g)$ is the smoothed counting function of the a-points, a-points of multiplicity k being counted as k points; $N_1(r, a; g)$ is the smoothed counting function of multiple a-points, in which an a-point of multiplicity k is counted $k-1$ times. If

$$\theta(a; g) = \liminf_{r\to\infty} (m(r, a; g)+N_1(r, a, g))/T(r, g);$$

then [6, Theorem 14.7.1, p. 230]

$$(1) \qquad \sum_a \theta(a; g) \leqq 2 \quad (g \text{ meromorphic}), \qquad \sum_a \theta(a; g) \leqq 1 \quad (g \text{ entire}).$$

Under the hypotheses of the theorem, if a is a simple zero of $f(z)$, then $g(z)-a$ has only a finite number of simple zeros, so that $2N_1(r, a; g) \geqq N(r, a; g)+O(\log r)$. Therefore

$$\begin{aligned} m(r, a; g)+N_1(r, a; g) &\geqq \tfrac{1}{2}(m(r, a; g)+N(r, a; g))+O(\log r) \\ &\geqq \tfrac{1}{2}T(r, g)+O(\log r). \end{aligned}$$

Since g is transcendental, $\log r = o(T(r, g))$ and so $\theta(a; g) \geqq \frac{1}{2}$ and the theorem follows from (1).

For our next result we shall need

LEMMA 2 (HAYMAN [5]). *If f is any transcendental meromorphic function and $g(z)$ is a transcendental entire function, then $T(r, f(g))/T(r, g) \to \infty$ as $r \to \infty$.*

REMARK. Though Hayman states Lemma 2 for entire functions f, it remains valid for meromorphic f as well.

THEOREM 3. *Let $Q(z)$ be a nonzero polynomial and let τ be a nonzero constant. If $F(z)$ is entire, transcendental of exponential type and for some constant c satisfies $F(z+\tau)-F(z) = Q(z)e^{cz}$, then $F(z)$ is pseudo-prime.*

[REMARK. The theorem is also true if $Q(z)$ is identically zero (see Theorem 9), but the proof is different.]

Proof. Assume that $F(z) = f(g(z))$ where f is transcendental and meromorphic and g is transcendental and entire.

We have $f(g(z+\tau))-f(g(z)) = Q(z)e^{cz}$. Therefore $g(z+\tau)-g(z)$ is an entire function with a finite number of zeros. By Lemma 2, $g(z)$ and so also $g(z+\tau)-g(z)$ is of order one, type zero, at most. Hence by Hadamard's factorization theorem

$g(z+\tau)-g(z)=$polynomial. Hence $g^{(n)}(z)$ is periodic for some integer n, and since it is at most of order 1 type zero it must be constant. It follows that $g(z)$ must be a polynomial.

DEFINITION. z_0 is said to be a fix-point of a function $f(z)$ if $f(z_0)=z_0$.

LEMMA 3 (ROSENBLOOM [10]). *If $P(z)$ is a nonlinear polynomial and $f(z)$ is entire transcendental, then $P(f(z))$ has infinitely many fix-points. Equivalently $f(P(z))$ also has infinitely many fix-points.*

From Theorem 3 and Lemma 3 we get

COROLLARY. *e^z+z is prime.*

This last result was stated by Rosenbloom [10] without proof.

THEOREM 4. *Let F (F^*) denote the family of entire (meromorphic) functions with at most a finite number of fix-points. Then* (i) *every entire function has at most one factorization $f(g(z))$, f transcendental, $f\in F$, g entire*; (ii) *every meromorphic function has at most two distinct factorizations $f_i(g_i(z))$, f_i meromorphic, not rational, $f_i\in F^*$, g_i entire.*

Proof. We prove the second part only. The proof of the first part is similar. Suppose $h(z)$ has three factorizations $f_i(g_i(z))$ $(i=1, 2, 3)$ of the type described in statement (ii). Assume that g_i are all distinct.

By Lemma 2, $T(r, g_i)=o(T(r, h))$. Hence, by a well-known theorem of Nevanlinna [7],

$$(1+o(1))T(r, h) \leqq \sum_{i=1}^{3} \bar{N}\left(r, \frac{1}{h-g_i}\right)$$

outside a set of r of finite linear measure, or

$$(1+o(1))T(r, h) \leqq \sum_{i=1}^{3} \bar{N}\left(r, \frac{1}{f_i(g_i)-g_i}\right) \tag{2}$$

outside a set of r of finite linear measure.

Suppose that each of the functions f_i, $i=1, 2, 3$ has at most a finite number of fix-points. Say f_i has fix-points z_{ij}, $j=1, 2, \ldots, K_i$. Then

$$\bar{N}\left(r, \frac{1}{f_i(g_i)-g_i}\right) \leqq \sum_{j=1}^{K_i} N\left(r, \frac{1}{g_i-z_{ij}}\right) \leqq cT(r, g_i) = o(T(r, h)), \tag{3}$$

where c is a constant.

Since (2) and (3) lead to a contradiction, our proof is complete.

Corollary 1 below is a generalization of the fact that $f_n(z)$, the nth iterate of $f(z)$, has infinitely many fix-points for $n>1$, a result also first proved by Rosenbloom. The corollary follows from Theorem 4 and the following lemma.

LEMMA 4 (BAKER [1]). *If $f(z)$ is a polynomial, then $P(f)=\{\text{entire } g; f(g(z))=g(f(z))\}$ contains transcendental functions if and only if $f(z)$ has one of the forms $f(z)=\text{const}$ or $f(z)=\gamma z+\delta$, δ and γ a root of unity.*

COROLLARY 1. *Let $f(z)$ be a transcendental entire function and let $g(z)$ be nonlinear and entire. If $f\not\equiv g$ and $f(g)=g(f)$, then one of f, g has infinitely many fix-points.*

Proof. By Lemma 4, g cannot be a nonlinear polynomial. By the above theorem, f or g must have infinitely many fix-points since they both are left factors of the entire function $f(g)$.

COROLLARY 2. *If f and g are transcendental entire, then f or $f(g)$ must have infinitely many fix-points.*

Corollary 2 was first proved by Rosenbloom [10].

COROLLARY 3. *If f is transcendental meromorphic and g and h are transcendental entire then one of $f(z)$, $f(g(z))$, $f(g(h(z)))$ has infinitely many fix-points.*

Proof of Corollary 2. Let $\eta(z)$ be nonlinear entire, then $f(g(\eta(z)))$ has the two transcendental left factors f and $f(g)$ and, consequently, one must have infinitely many fix-points.

The proof of Corollary 3 is similar.

COROLLARY 4. *If f is a meromorphic (an entire) periodic function and $g(z)$ is entire, then $f(f(g))$ has infinitely many fix-points.*

Proof. Let $f\in F^*$. Suppose f is periodic with period τ. For any entire g, $f(g(z)+n\tau)=f(g(z))$ $(n=1, 2, 3, \ldots)$. This implies that f is not rational and the assertion follows from Theorem 4 (ii). Thus the corollary follows for f meromorphic. When f is entire periodic and g is entire, then $g(f)$ is periodic and must have infinitely many fix-points. It follows that $f(g)$ must have infinitely many fix-points.

COROLLARY 5. *If $f(g)$ is periodic and f is meromorphic with at most finitely many fix points, then g is periodic.*

Proof. Let $f\in F^*$. Suppose g is entire and $f(g(z))$ is periodic with period τ. We have $f(g(z+n\tau))=f(g(z))$, $n=1, 2, \ldots$. Again it follows from the last part of Theorem 4 that $g(z+n\tau)=g(z+m\tau)$ for some n and m with $n\not\equiv m$.

This generalizes a previous result of the author [4]. If $e^{g(z)}+g(z)$ is periodic for an entire function g, then g must be periodic.

It follows from the above discussion that if f is an entire function such that $f(f(z))$ is periodic, then f has infinitely many fix-points. An interesting related problem which the author has not been able to resolve is whether $f(f(z))$ is periodic if and only if $f(z)$ is. Another problem of this type is the following:

Let f be an entire function. How many entire solutions, g, does the functional

equation $ff=gg$ have([2])? The methods of this paper do not seem to work for these problems. We do have, however,

COROLLARY 6. *Let f and g be entire functions and let $f_n(z)$ denote the nth iterate of $f(z)$. If for some integer $n>1$, $f_n(z)=ag_n(z)+b$, then either $f=cg+d$ for some constants c and d, and f and g are polynomials or one of f, g has infinitely many fix-points.*

Proof. When f and g are transcendental, the assertion follows at once from Theorem 4 (i). If f and g are polynomials they must be of the same degree, as comparison of the highest powers of $f_n(z)$ and $ag_n(z)+b$ shows. Therefore f_{n-1} and g_{n-1} are also of the same degree and it is possible to choose the number λ so that $L=f_{n-1}-\lambda g_{n-1}$ is of degree lower than the degree of g_{n-1}. We show that L is a constant. The theorem then follows by induction on n.

Suppose that L is of degree m and that g_{n-1} is of degree $h>m$. If

$$f = A_0+A_1z+\cdots+A_kz^k, \qquad A_k \neq 0$$

and

$$g = B_0+B_1z+\cdots+B_kz^k, \qquad B_k \neq 0,$$

then $f(\lambda g_{n-1}+L)=ag(g_{n-1})+b$. Comparison of the terms of degree $k\cdot h$ yields $A_k\lambda^k=aB_k$. After cancellation of the g_{n-1}^k-terms, the highest power of z on the right-hand side of the equation is $\leqq(k-1)h$; the highest power on the left is $(k-1)h+m$. Therefore $m=0$, L is a constant.

We now give a generalization of a theorem of Rényi [9] and the author [3].

THEOREM 5. *Let $f(z)$ and $h(z)$ be arbitrary nonconstant meromorphic functions. The functional equation $f(g)=h$ has at most a denumerable number of solutions g.*

Proof. For given w_0 the number of solutions $g(z)$ of $f(g(z))=h(z)$, $g(z_0)=w_0$ is finite by the inverse function theorem for entire functions. The possible values of w_0 must satisfy $f(w_0)=h(z_0)$. This gives a finite or denumerable set of w_0.

The same proof also shows:

COROLLARY 1. *For any rational function $P(w)$ and any meromorphic function $h(z)$ the functional equation $P(f(z))=h(z)$ has at most a finite number of solutions $f(z)$.*

COROLLARY 2 (RÉNYI [9] AND THE AUTHOR [3]). *For any polynomial P and any entire function $f(z)$, $P(f(z))$ is periodic if and only if $f(z)$ is.*

COROLLARY 3. *Let f be an entire function. If $f(z_0+n)=f(z_0)$ for an infinite number of integers n and some complex number z_0 and if for some meromorphic function, g, $g(f)$ is periodic with period 1, then f is periodic.*

([2]) Subsequent to the completion of this paper, I. N. Baker and the author showed that there are at most denumerably many such g.

Theorem 5 leads to an interesting conjecture. For any factorization of a meromorphic function h, say

$$h = f(g), \tag{4}$$

$$h = fL(L^{-1}g) \tag{5}$$

is another such factorization, where L is a linear transformation. The only meromorphic functions with meromorphic inverses are the linear transformations. It is, therefore, reasonable to expect that if one considers such factorizations of h as (4) and (5) equivalent, then

CONJECTURE. Any meromorphic function has at most denumerably many non-equivalent factorizations.

THEOREM 6. *Let $g(z)=u(z)+iv(z)$ be entire. Let $f(z)$ be entire and periodic with real period. If $f(g(z))$ is periodic with real period and $v(z)$ is bounded on some horizontal half line L, then $g(z)=P(z)+Hz$, where $P(z)$ is periodic with real period and H is a real constant.*

Proof. Without loss of generality we may suppose that L is the positive x-axis. Let $f(w)$ have period $a>0$, $f(g(z))$, period $b>0$. Every point $g(mb)$ (m a positive integer) is congruent (mod a) to a point Z_m in $0\leqq x<a$, $|y|<K$. Also $f\{g(mb)\}=f(Z_m)$. If the point-set $\{Z_m\}$ is infinite, it has a limit point in $0\leqq x\leqq a$, $|y|\leqq K$ and therefore $f(z)\equiv$ constant. If f is not constant, then the point set Z_m is finite, and there is an infinite set M of m such that there is a Z and an integer K_m with $g(mb)-k_ma=Z$, $(m\in M)$.

The equation

$$f(\gamma(z)) = f(g(z)), \qquad \gamma(0) = Z \tag{6}$$

has the solutions $\gamma(z)=g(z+mb)-K_ma$, $(m\in M)$. But (6) has at most a finite number of solutions (see the beginning of the proof of Theorem 5), so that

$$g(z+m_1b)-K_1a = g(z+m_2b)-K_2a.$$

This proves the theorem with $H=-(K_2-K_1)a/(m_2-m_1)b$.

As a further generalization of Corollary 2 of Theorem 5 we have

THEOREM 7. *If f is any entire function of order less than $\frac{1}{2}$ and g is entire, then $f(g)$ is periodic if and only if g is.*

Proof. Let $F(z)=f(g(z))$ and suppose that $F(z+\tau)=F(z)$. Let L be the line $z_0+\lambda\tau$, $-\infty<\lambda<\infty$. The periodic function $F(z)$ is bounded on L. If $g(z)$ is unbounded on L, then $g(L)$ is a path extending arbitrarily far from the origin on which $f(z)$ is bounded. This, however, is impossible, since by a well-known theorem of Wiman any entire function of order $\frac{1}{2}$ must be unbounded on every curve going to infinity. It follows that $g(z)$ is bounded on L. Choose a value z_0 on L such that $\alpha=f(g(z_0))$ is not an algebraic singularity of $f_{-1}(z)$, the inverse function of $f(z)$.

Now $\{g(z_0+n\tau)\}$, $n=1, 2, \ldots$ is bounded, say $|g(z_0+n\tau)| \leqq M$, while $f(g(z_0+n\tau))=f(g(z_0))=\alpha$. Thus all $g(z_0+n\tau)$ are among the finite set of solutions of $f(w)=\alpha$ which belong to $|w| \leqq M$. Hence for some $m \neq n$, $g(z_0+m\tau)=g(z_0+n\tau)$. Moreover, for all small ε, $f(g(z_0+\varepsilon+m\tau))=f(g(z_0+\varepsilon+n\tau))=\beta(\varepsilon)$, so that $g(z_0+\varepsilon+m\tau)$ and $g(z_0+\varepsilon+n\tau)$ are both equal to the unique root of $f(w)=\beta(\varepsilon)$ which lies near $g(z_0+m\tau)=g(z_0+n\tau)$. Thus we must have $g(z+m\tau)\equiv g(z+n\tau)$, and $g(z)$ has period $(m-n)\tau$.

Wiman's theorem mentioned in the proof can be generalized to lower order (see Whittaker [12]). Thus we have

THEOREM 7A. *If f is any entire function of lower order less than $\frac{1}{2}$ and g is entire, then $f(g)$ is periodic if and only if g is.*

NOTE. The function $\cos z$ illustrates that $\frac{1}{2}$ is the best upper bound in the above theorem.

Earlier we asked the question whether for an entire function f, ff is periodic if and only if f is. More generally one can ask:

If f and g are entire and $f(g)=g(f)=F$ is periodic, then can one expect that f and g are periodic? In other words, if f or g is nonperiodic can F be periodic?

Theorem 7A yields the following partial answer.

COROLLARY. *If f and g are entire functions, not both periodic, which commute and $F=f(g)$ has the property that for some $\varepsilon>0$, $M_F(r)<\exp(\exp(r^{1/2-\varepsilon}))$ for an infinite sequence of r approaching infinity, then F cannot be periodic.*

LEMMA 5 (PÓLYA [8]). *If $\phi(z)$, $g(z)$ and $h(z)$ are entire functions such that, $\phi(z)=g(h(z))$ and $h(0)=0$, then there is a positive constant c, independent of $g(z)$, $h(z)$ and r with $M_\phi(r)>M_g\{cM_h\{r/2\}\}$, where $M_f(r)=\max_{|z|=r}|f(z)|$.*

Proof of corollary. One can, after some simple transformations, apply Lemma 5 to F and conclude that either f or g must be of lower order less than $\frac{1}{2}$. Thus by the above theorem F cannot be periodic.

From the arguments of Rosenbloom [10] one can conclude that e^z is pseudo-prime. This also follows from Theorem 1. More generally we have

THEOREM 9. *Every periodic entire function of exponential type is pseudo-prime.*

Proof of Theorem. Let $F=f(g)$ satisfy the hypotheses of the theorem.

By Lemma 1 either g is a polynomial or f is of order zero. If f is of order zero, g must be periodic, by Theorem 7. But then $g(z)$ has to be of order $\geqq 1$ which by Lemma 2 contradicts the fact that $f(g(z))$ is of exponential type, unless f is a polynomial.

We conclude this paper with

THEOREM 10. *A meromorphic function and its derivative cannot have a common right factor other than one of the form $e^{cz+b}+d$ where c, b and d are constants.*

Proof. Suppose that $h=f(g)$ is meromorphic and $h'=l(g)$. Then $g'(z)f'(g(z))=l(g(z))$.

Letting $H(w)=l(w)/f'(w)$ we have $g'=H(g)$.

Since $T(r, g')<(2+o(1))T(r, g)$ as $r\to\infty$, it follows from Lemma 2 that $H(w)=P(w)/Q(w)$, where $P(w)$, $Q(w)$ are relatively prime polynomials.

Thus for some constant c we have

$$(7)\qquad C(g-a_1)^{n_1}(g-a_2)^{n_2}\cdots(g-a_k)^{n_k} = g'(g-b_1)^{m_1}(g-b_2)^{m_2}\cdots(g-b_t)^{m_t},$$

where a_j and b_j are distinct complex numbers and the n's and m's positive integers. Each a_j and each b_j are Picard values of $g(z)$, as can be seen by considering the order to which each side of (7) vanishes at a root of $g(z)=a_j$ (or $g(z)=b_j$). Since there can only be one finite Picard value, the equation must be of the form $c(g-d)^k=g'$, where k is an integer.

Elementary integrations show that the only solutions $g(z)$ are of the desired form.

Added in Proof. After this paper was completed, the author discovered that Theorem 7 has already been proved by I. N. Baker. (See *On some results of A. Rényi and C. Rényi concerning periodic entire functions*, Acta Sci. Math. (Szeged) **27** (1966) 197–200.)

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